Fundamental solutions of evolutionary PDOs and rapidly decreasing distributions

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Abstract

Let $P(\partial_0, \partial_1, \ldots, \partial_n)$ be a PDO on $\mathbb{R}^{1+n}$ with constant coefficients. It is proved that

(i) the real parts of the λ-roots of the polynomial $P(\lambda, i\xi_1, \ldots, i\xi_n)$ are bounded from above when $(\xi_1, \ldots, \xi_n)$ ranges over $\mathbb{R}^n$

if and only if

(ii) P has a fundamental solution with support in $H_+ = \{(x_0, x_1, \ldots, x_n) \in \mathbb{R}^{1+n} : x_0 \geq 0\}$ having some special properties expressed in terms of the L. Schwartz space $\mathcal{O}'_C$ of rapidly decreasing distributions.

Moreover, it is proved that the fundamental solution with support in H_+ having these special properties is unique.

1 Introduction and the main result

1.1 Rapidly decreasing distributions

By Theorem IX in Sec. VII.5 of L. Schwartz's book [S2], for every distribution $T \in \mathcal{D}'(\mathbb{R}^n)$ the following two conditions are equivalent:

(1.1) $T * \varphi \in \mathcal{S}(\mathbb{R}^n)$ for every $\varphi \in \mathcal{D}(\mathbb{R}^n)$,

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(1.2) for every $k \in \mathbb{N}_0$ there is $m_k \in \mathbb{N}_0$ such that $T = \sum_{|\alpha| \le m_k} \partial^\alpha F_{k,\alpha}$ where, for every multiindex $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}_0$ of length $|\alpha| = \alpha_1 + \dots + \alpha_n \le m_k$, $F_{k,\alpha}$ is a continuous function on $\mathbb{R}^n$ such that $\sup_{x \in \mathbb{R}^n} (1 + |x|)^k |F_{k,\alpha}(x)| < \infty$.

In the above, and everywhere in the following, $\partial^\alpha = \partial_1^{\alpha_1} \dots \partial_n^{\alpha_n}$ where $\partial_1, \dots, \partial_n$ are partial derivatives of the first order not multiplied by any factor. Each of the conditions (1.1), (1.2) is satisfied if and only if the distribution T is *rapidly decreasing*, where the definition of rapid decrease, due to L. Schwartz, refers to the notion of boundedness of a distribution. The space of rapidly decreasing distributions on $\mathbb{R}^n$ is denoted by $\mathcal{O}'_C(\mathbb{R}^n)$. From (1.2) it follows that

(1.3) whenever $T \in \mathcal{O}'_C(\mathbb{R}^n)$ and $\varphi \in C_b^\infty(\mathbb{R}^n)$, then $\varphi T \in \mathcal{O}'_C(\mathbb{R}^n)$.

It is clear from (1.2) that $\mathcal{O}'_C(\mathbb{R}^n) \subset \mathcal{S}'(\mathbb{R}^n)$, so that the Fourier transform $\mathcal{F}T$ makes sense for every $T \in \mathcal{O}'_C(\mathbb{R}^n)$. By Theorem XV in Sec. VII.8 of [S2],

$$\mathcal{F}\mathcal{O}'_C(\mathbb{R}^n) = \mathcal{O}_M(\mathbb{R}^n), \tag{1.4}$$

where $\mathcal{O}_M(\mathbb{R}^n)$ denotes the space of *infinitely differentiable slowly increasing functions* on $\mathbb{R}^n$. Recall that $\phi \in \mathcal{O}_M(\mathbb{R}^n)$ if and only if $\phi \in C^\infty(\mathbb{R}^n)$ and for every $\alpha \in \mathbb{N}_0^n$ there is $m_\alpha \in \mathbb{N}_0$ such that

$$\sup_{\xi \in \mathbb{R}^n} (1 + |\xi|)^{-m_\alpha} |\partial^\alpha \phi(\xi)| < \infty.$$

Complete proofs of theorems about $\mathcal{O}'_C(\mathbb{R}^n)$ and $\mathcal{O}_M(\mathbb{R}^n)$ needed in the present paper may be found in [K].

1.2 The main result

Our object of interest will be the differential operator $P(\partial_0, \partial_1, \dots, \partial_n)$ on $\mathbb{R}^{1+n} = \{(x_0, x_1, \dots, x_n) : x_\nu \in \mathbb{R} \text{ for } \nu = 0, \dots, n\}$ with constant coefficients, and the associated polynomial $P(\lambda, i\xi_1, \dots, i\xi_n)$ defined on $\mathbb{C} \times \mathbb{R}^n$. A distribution N on $\mathbb{R}^{1+n}$ such that

$$PN \equiv \delta$$

is called a *fundamental solution* for the operator P. Let

$$H_+ = \{(x_0, x_1, \dots, x_n) \in \mathbb{R}^{1+n} : x_0 \ge 0\}.$$

If there exists a fundamental solution N for P such that $\operatorname{supp} N \subset H_+$, then the operator P is said to be *evolutionary* with respect to H_+. For every fixed $\lambda \in \mathbb{C}$ let $e_{-\lambda}$ be the function on $\mathbb{R}^{1+n}$ given by $e_{-\lambda}(x_0, x_1, \ldots, x_n) = \exp(-\lambda x_0)$ for $(x_0, x_1, \ldots, x_n) \in \mathbb{R}^{1+n}$. For $\vartheta \in \mathcal{D}(\mathbb{R})$, denote by ϑ_0 the function on $\mathbb{R}^{1+n}$ such that $\vartheta_0(x_0, x_1, \ldots, x_n) = \vartheta(x_0)$. Let

$$\mathcal{O}'_{\mathrm{LOC}}(H_+) = \{T \in \mathcal{D}'(\mathbb{R}^{1+n}) : \operatorname{supp} T \subset H_+,\\ \vartheta_0 T \in \mathcal{O}'_C(\mathbb{R}^{1+n}) \text{ for every } \vartheta \in \mathcal{D}(\mathbb{R})\}.$$

Theorem. *Let $P(\partial_0, \partial_1, \ldots, \partial_n)$ be the differential operator on $\mathbb{R}^{1+n}$ with constant coefficients. Let*

$$\omega_0 = \sup\{\operatorname{Re}\lambda : \lambda \in \mathbb{C} \textit{ and there is } (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n\\ \textit{such that } P(\lambda, i\xi_1, \ldots, i\xi_n) = 0\}.$$

Then the following two conditions are equivalent:

(i) $\omega_0 < \infty$,

(ii) *the differential operator $P(\partial_0, \partial_1, \ldots, \partial_n)$ has a fundamental solution N belonging to $\mathcal{O}'_{\mathrm{LOC}}(H_+)$.*

Furthermore, if (i) *and* (ii) *are satisfied, then the fundamental solution N as in* (ii) *is unique and satisfies*

(iii) $\omega_0 = \inf\{\operatorname{Re}\lambda : \lambda \in \mathbb{C},\ e_{-\lambda}N \in \mathcal{O}'_C(\mathbb{R}^{1+n})\}$, *and* $e_{-\lambda}N \in \mathcal{O}'_C(\mathbb{R}^{1+n})$ *whenever* $\operatorname{Re}\lambda > \omega_0$.

1.3 Remarks

Condition (i) can be called the Petrovskiĭ condition because it first appeared in I. G. Petrovskiĭ's paper [P]. Namely, in [P], in the footnote on p. 24, it was conjectured that, if the polynomial $P(\lambda, i\xi_1, \ldots, i\xi_n)$ is unital with respect to λ, then this condition is equivalent to a certain formally weaker condition also concerning the λ-roots of $P(\lambda, i\xi_1, \ldots, i\xi_n)$. The validity of this conjecture was proved by L. Gårding in [G]. I. G. Petrovskiĭ noticed the significance of smooth slowly increasing functions for the theory of evolutionary PDEs with constant coefficients. L. Schwartz explained in [S1] how the results of Petrovskiĭ may be elucidated by placing them in the framework of rapidly decreasing distributions and smooth slowly increasing functions. (Condition (i) was not mentioned in [S1]; notice that [S1] was earlier than [G].)

L. Hörmander proved in [H1] that if $P(\zeta_0, \zeta_1, \dots, \zeta_n)$ is a polynomial of $1+n$ complex variables, then the following two conditions are equivalent:

(i)* there are constants $A \in]-\infty, \infty[$ and $r \in]0, \infty[$ such that

$$\inf\{\operatorname{Re} F(\zeta_1, \dots, \zeta_n) : (\zeta_1, \dots, \zeta_n) \in B_{i\xi_1,\dots,i\xi_n;r}\} \le A$$

for every $(\xi_1, \dots, \xi_n) \in \mathbb{R}^n$ and every function F holomorphic in the ball

$$B_{i\xi_1,\dots,i\xi_n;r} = \Big\{(\zeta_1, \dots, \zeta_n) \in \mathbb{C}^n : \sum_{\nu=1}^{n} |\zeta_\nu - i\xi_\nu|^2 < r^2\Big\}$$

such that $P(F(\zeta_1, \dots, \zeta_n), \zeta_1, \dots, \zeta_n) = 0$ in $B_{i\xi_1,\dots,i\xi_n;r}$,

(ii)* the differential operator $P(\partial_0, \partial_1, \dots, \partial_n)$ has a fundamental solution with support in H_+.

The equivalence (i)$^*\Leftrightarrow$(ii)* was reproved in Sec. 12.8 of [H2]. The fundamental solution occurring in (ii)* need not be unique. It is non-unique if (i)* holds and the boundary of H_+ is characteristic for $P(\partial_0, \partial_1, \dots, \partial_n)$. Obviously (i) implies (i)*. Furthermore, as indicated in [H1], the operator $\partial_0 - i(\partial_1 + 1)^2$ satisfies (i)* but does not satisfy (i). Therefore condition (i)* is essentially weaker than (i).

Let us stress that in [H1], and in the present paper, the largest power of λ in $P(\lambda, i\xi_1, \dots, i\xi_n)$ is multiplied by a polynomial of $\xi_1, \dots, \xi_n$ which, in contrast to the assumption (5) in Sec. 3.10 of [R], may vanish for some $(\xi_1, \dots, \xi_n) \in \mathbb{R}^n$.

2 Existence of a fundamental solution satisfying (ii) and (iii)

2.1 Application of the Tarski–Seidenberg theorem

We are going to prove that if (i) holds, then the differential operator $P(\partial_0, \partial_1, \dots, \partial_n)$ has a fundamental solution N satisfying the conditions (ii) and (iii). So, suppose that (i) holds and let

$$\mathcal{N} = \{(\sigma, \xi_0, \dots, \xi_n) \in \mathbb{R}^{2+n} : P(\sigma + i\xi_0, i\xi_1, \dots, i\xi_n) = 0\}.$$

Then $\mathcal{N} \subset \{(\sigma, \xi_0, \dots, \xi_n) \in \mathbb{R}^{2+n} : \sigma \le \omega_0\}$, and hence, by Theorem A.3 from the Appendix to [T] or by Theorem 3.2 of [Go]$^{*)}$, there are $c, \mu, \mu' \in$

$^{*)}$Following the idea of L. Hörmander, these theorems are deduced from the Tarski–Seidenberg theorem about projections of semi-algebraic sets.

$]0,\infty[$ such that whenever $\sigma \in]\omega_0,\infty[$, and $(\xi_0,\dots,\xi_n) \in \mathbb{R}^{1+n}$, then

$$\begin{aligned}(2.1)\quad |P(\sigma+i\xi_0,i\xi_1,\dots,i\xi_n)| &\geq c(\operatorname{dist}((\sigma,\xi_0,\dots,\xi_n);\mathcal{N}))^{\mu} \\ &\quad\cdot (1+(\sigma^2+\xi_0^2+\cdots+\xi_n^2)^{1/2})^{-\mu'} \\ &\geq c(\sigma-\omega_0)^{\mu}(1+|\sigma+i\xi_0|+(\xi_1^2+\cdots+\xi_n^2)^{1/2})^{-\mu'}.\end{aligned}$$

2.2 The slowly increasing functions $\widehat{N_\sigma}$ and the rapidly decreasing distributions N_σ

For every $\sigma \in]\omega_0,\infty[$ let

$$(2.2)\qquad \widehat{N_\sigma}(\xi_0,\dots,\xi_n) = (P(\sigma+i\xi_0,i\xi_1,\dots,i\xi_n))^{-1}$$

for $(\xi_0,\dots,\xi_n) \in \mathbb{R}^{1+n}$. Then, for every multiindex $\alpha \in \mathbb{N}^{1+n}$,

$$\partial^\alpha \widehat{N_\sigma}(\xi_0,\dots,\xi_n) = (P(\sigma+i\xi_0,i\xi_1,\dots,i\xi_n))^{-1-|\alpha|}Q_\alpha(\sigma,\xi_0,\dots,\xi_n)$$

where Q_α is a polynomial. Consequently, (2.1) implies that

$$(2.3)\qquad \widehat{N_\sigma} \in \mathcal{O}_M(\mathbb{R}^{1+n}) \quad \text{for every } \sigma \in]\omega_0,\infty[.$$

Let

$$(2.4)\qquad N_\sigma = \mathcal{F}^{-1}\widehat{N_\sigma}$$

where $\mathcal{F}$ denotes the Fourier transformation on $\mathbb{R}^{1+n}$ such that

$$\begin{aligned}(\mathcal{F}\varphi)(\xi_0,\dots,\xi_n) &= \widehat{\varphi}(\xi_0,\dots,\xi_n) \\ &= \underset{\mathbb{R}^{1+n}}{\int\cdots\int} e^{-i\sum_{\nu=0}^{n} x_\nu\xi_\nu}\varphi(x_0,\dots,x_n)\,dx_0\dots dx_n\end{aligned}$$

for $\varphi \in \mathcal{S}(\mathbb{R}^{1+n})$, and $\mathcal{F}$ is extended onto $\mathcal{S}'(\mathbb{R}^{1+n})$ by duality. From (1.4) and (2.3) it follows that

$$(2.5)\qquad N_\sigma \in \mathcal{O}'_C(\mathbb{R}^{1+n}) \quad \text{for every } \sigma \in]\omega_0,\infty[.$$

Furthermore, from (2.2) it follows that

(2.6) if $\sigma \in]\omega_0,\infty[$ then N_σ is a fundamental solution for the differential operator $P(\sigma+\partial_0,\partial_1,\dots,\partial_n)$.

Take $\sigma \in]\omega_0, \infty[$, and consider the distribution $e_\sigma N_\sigma \in \mathcal{D}'(\mathbb{R}^{1+n})$. By the Parseval equality, for every $\varphi \in \mathcal{D}(\mathbb{R}^{1+n})$ one has

$$\begin{aligned}&\langle e_\sigma N_\sigma, \varphi\rangle = \langle N_\sigma, e_\sigma\varphi\rangle = (2\pi)^{-1-n}\langle \widehat{N_\sigma}, \widehat{e_\sigma\varphi}^{\vee}\rangle \\ &= (2\pi)^{-1-n} \underset{\mathbb{R}^{1+n}}{\int\cdots\int} (\widehat{e_\sigma\varphi}(-\xi_0, \ldots, -\xi_n)(P(\sigma + i\xi_0, i\xi_1, \ldots, i\xi_n))^{-1}\, d\xi_0 \ldots d\xi_n.\end{aligned}$$

For every $\varphi \in \mathcal{D}(\mathbb{R}^{1+n})$ the Fourier integral

$$\widehat{\varphi}(\zeta_0, \ldots, \zeta_n) = \underset{\mathbb{R}^{1+n}}{\int\cdots\int} e^{-i\sum_{\nu=0}^{n} x_\nu\zeta_\nu}\varphi(x_0, \ldots, x_n)\, dx_0 \ldots dx_n$$

makes sense for $(\zeta_0, \ldots, \zeta_n) \in \mathbb{C}^{1+n}$ and defines the holomorphic extension of $\widehat{\varphi}$ from $\mathbb{R}^{1+n}$ onto $\mathbb{C}^{1+n}$. This holomorphic extension satisfies

$$\widehat{e_\sigma\varphi}(\zeta_0, \ldots, \zeta_n) = \widehat{\varphi}(\zeta_0 + i\sigma, \zeta_1, \ldots, \zeta_n).$$

Consequently, whenever $\varphi \in \mathcal{D}(\mathbb{R}^{1+n})$ and $\sigma \in]\omega_0, \infty[$, then

$$\begin{aligned}\text{(2.7)}\qquad \langle e_\sigma N_\sigma, \varphi\rangle = (2\pi)^{-1-n} \underset{\mathbb{R}^{1+n}}{\int\cdots\int} &\widehat{\varphi}(-\xi_0 + i\sigma, -\xi_1, \ldots, -\xi_n) \\ &\cdot P(\sigma + i\xi_0, \xi_1, \ldots, \xi_n)^{-1}\, d\xi_0 \ldots d\xi_n.\end{aligned}$$

Integration by parts shows that whenever $\varphi \in \mathcal{D}(\mathbb{R}^{1+n})$ and $l \in \mathbb{N}$, then

$$\begin{aligned}\text{(2.8)}\qquad &(1 + |\xi_0 - i\sigma|^l + |\xi_1|^l + \cdots + |\xi_n|^l)|\widehat{\varphi}(-\xi_0 + i\sigma, -\xi_1, \ldots, -\xi_n)| \\ &\leq \Big(\|\varphi\|_{L^1(\mathbb{R}^{1+n})} + \sum_{\nu=0}^{n} \|\partial_\nu^l\varphi\|_{L^1(\mathbb{R}^{1+n})}\Big) \exp(H_\varphi(\sigma))\end{aligned}$$

for every $\sigma, \xi_0, \ldots, \xi_n \in \mathbb{R}$ where

$$\text{(2.9)}\qquad H_\varphi(\sigma) = \sup\{\sigma x_0 : (x_0, \ldots, x_n) \in \operatorname{supp}\varphi\}.$$

From (2.1), (2.7)–(2.9) and the Cauchy integral theorem it follows that

(2.10) the distribution $e_\sigma N_\sigma \in \mathcal{D}'(\mathbb{R}^{1+n})$ does not depend on σ provided that $\sigma \in]\omega_0, \infty[$,

(2.11) $\lim_{\sigma\to\infty}\langle e_\sigma N_\sigma, \varphi\rangle = 0$ whenever $\varphi \in \mathcal{D}(\mathbb{R}^{1+n})$ and $\operatorname{supp}\varphi \subset \mathbb{R}^{1+n}\setminus H_+$.

2.3 The fundamental solution N

Thanks to (2.10) we may define the distribution $N \in \mathcal{D}'(\mathbb{R}^{1+n})$ by the equality

$$N = e_\sigma N_\sigma \quad \text{for every } \sigma \in \,]\omega_0, \infty[. \tag{2.12}$$

From (2.11) it follows that

$$\operatorname{supp} N \subset H_+. \tag{2.13}$$

For every $\sigma \in \mathbb{R}$ let

$$S_\sigma = P(\sigma + \partial_0, \partial_1, \dots, \partial_n)\delta. \tag{2.14}$$

Since $(-\partial_0)^k (e_{-\sigma}\varphi) = e_{-\sigma}(\sigma - \partial_0)^k \varphi$, it follows that

$$P(-\partial_0, -\partial_1, \dots, -\partial_n)(e_{-\sigma}\varphi) = e_{-\sigma} P(\sigma - \partial_0, -\partial_1, \dots, -\partial_n)\varphi \tag{2.15}$$

for every $\sigma \in \mathbb{R}$ and $\varphi \in \mathcal{D}(\mathbb{R}^{1+n})$. From (2.15) one infers that

$$\begin{aligned}
\langle S_0, e_{-\sigma}\varphi \rangle &= [P(-\partial_0, -\partial_1, \dots, -\partial_n)(e_{-\sigma}\varphi)](0) \\
&= [e_\sigma P(-\partial_0, -\partial_1, \dots, -\partial_n)(e_{-\sigma}\varphi)](0) \\
&= [P(\sigma - \partial_0, -\partial_1, \dots, -\partial_n)\varphi](0) = \langle S_\sigma, \varphi \rangle,
\end{aligned}$$

proving that

$$S_\sigma = e_{-\sigma} S_0 \quad \text{for every } \sigma \in \mathbb{R}. \tag{2.16}$$

From (2.6), (2.12) and (2.15) it follows that whenever $\sigma \in \,]\omega_0, \infty[$, then

$$PN = S_0 * N = (e_\sigma S_\sigma) * (e_\sigma N_\sigma) = e_\sigma (S_\sigma * N_\sigma) = e_\sigma \delta = \delta,$$

so that

(2.17) $\quad N$ is a fundamental solution for the operator P.

Above we have used the fact that whenever $T, U \in \mathcal{D}'(\mathbb{R}^{1+n})$, $\sigma \in \mathbb{R}$, and one of T, U has compact support, then $e_\sigma(T * U) = (e_\sigma T) * (e_\sigma U)$. This is true under the additional assumption that $T, U \in L^1_{\mathrm{loc}}(\mathbb{R}^{1+n})$, and this case implies the general assertion by regularization.

2.4 Properties of N

If $\vartheta \in \mathcal{D}(\mathbb{R})$ and $\sigma \in]\omega_0, \infty[$, then $\vartheta_0 e_\sigma$ is bounded on $\mathbb{R}^{1+n}$ together with all its partial derivatives, so that, by (1.3), $\vartheta_0 N = (\vartheta_0 e_\sigma)N_\sigma \in \mathcal{O}'_C(\mathbb{R}^{1+n})$ because $N_\sigma \in \mathcal{O}'_C(\mathbb{R}^{1+n})$. Hence, by (2.13),

$$N \in \mathcal{O}'_{\mathrm{LOC}}(H_+). \tag{2.18}$$

The relations (2.17) and (2.18) show that (i) implies (ii). We are going to prove that N defined by (2.12) satisfies (iii). To this end, take $\lambda \in \mathbb{C}$ such that $\operatorname{Re}\lambda \in]\omega_0, \infty[$. Let $\sigma = \frac{1}{2}(\omega_0 + \operatorname{Re}\lambda)$. Then $e_{-\lambda}N = e_{\sigma-\lambda}N_\sigma \in \mathcal{O}'_C(\mathbb{R}^{1+n})$ because $N_\sigma \in \mathcal{O}'_C(\mathbb{R}^{1+n})$, $\operatorname{supp} N_\sigma \subset H_+$, and $e_{\sigma-\lambda}$ is bounded together with all its partial derivatives on the set $\{(x_0, \dots, x_n) \in \mathbb{R}^{1+n} : x_0 > -1\}$. It remains to prove that

$$\text{if } \lambda \in \mathbb{C} \text{ and } e_{-\lambda}N \in \mathcal{O}'_C(\mathbb{R}^{1+n}), \text{ then } \operatorname{Re}\lambda \geq \omega_0. \tag{2.19}$$

So, suppose that $\lambda \in \mathbb{C}$ and $e_{-\lambda}N \in \mathcal{O}'_C(\mathbb{R}^{1+n})$. Take any $\sigma \in]\operatorname{Re}\lambda, \infty[$. Since $e_{\lambda-\sigma}$ is bounded on $\{(x_0, \dots, x_n) \in \mathbb{R}^{1+n} : x_0 > -1\}$ together with all its partial derivatives, it follows by (1.3) that $e_{-\sigma}N = e_{\lambda-\sigma}(e_{-\lambda}N) \in \mathcal{O}'_C(\mathbb{R}^{1+n})$. Furthermore

$$S_\sigma * (e_{-\sigma}N) = (e_{-\sigma}S_0) * (e_{-\sigma}N) = e_{-\sigma}(S_0 * N) = e_{-\sigma}\delta = \delta.$$

Let $\phi = \mathcal{F}(e_{-\sigma}N)$. Then $\phi \in \mathcal{O}_M(\mathbb{R}^{1+n})$ and

$$P(\sigma + i\xi_0, i\xi_1, \dots, i\xi_n) \cdot \phi(\xi_0, \dots, \xi_n) = [\mathcal{F}(S_\sigma * (e_{-\sigma}N))](\xi_0, \dots, \xi_n) = 1$$

for every $(\xi_0, \dots, \xi_n) \in \mathbb{R}^{1+n}$. This implies that $P(\sigma + i\xi_0, i\xi_1, \dots, i\xi_n) \neq 0$ for every $(\xi_0, \dots, \xi_n) \in \mathbb{R}^{1+n}$. Since this is true for every $\sigma \in]\operatorname{Re}\lambda, \infty[$, it follows that $\operatorname{Re}\lambda \geq \omega_0$, proving (2.19).

3 Uniqueness of the fundamental solution belonging to $\mathcal{O}'_{\mathrm{LOC}}(H_+)$

3.1 An associativity relation for convolution

Lemma 3.1. *Suppose that* (i) *holds. Fix $\sigma \in]\omega_0, \infty[$ and define N_σ and S_σ by* (2.4) *and* (2.14). *Suppose moreover that $U \in \mathcal{O}'_{\mathrm{LOC}}(H_+)$ and that $P(\partial_0, \partial_1, \dots, \partial_n)U$ has compact support. Then*

$$(N_\sigma * S_\sigma) * (e_{-\sigma}U) = N_\sigma * (S_\sigma * (e_{-\sigma}U)). \tag{3.1}$$

Proof. Notice that both sides of (3.1) are well defined because every sign $*$ in (3.1) denotes a convolution of two distributions on $\mathbb{R}^{1+n}$ one of which has compact support. To see this it is sufficient to observe that $\operatorname{supp} S_\sigma = \{0\}$, $N_\sigma * S_\sigma = S_\sigma * N_\sigma = \delta$, and, by (2.15),

$$S_\sigma * (e_{-\sigma}U) = P(\sigma + \partial_0, \partial_1, \dots, \partial_n)(e_{-\sigma}U) = e_{-\sigma}(P(\partial_0, \partial_1, \dots, \partial_n)U)$$

has compact support. However, from the three factors N_σ, S_σ and $e_{-\sigma}U$ occurring in (3.1) only one has compact support, so that (3.1) does not follow from any of the simple criterions of the associativity of convolution. In order to prove that both sides of (3.1) are equal we will apply an argument going back to C. Chevalley ([Che, pp. 120–121], proof of Theorem 2.2) which reduces the problem to the Fubini–Tonelli theorem.

Since the set $\{\varphi_1 * \varphi_2 * \varphi_3 : \varphi_i \in \mathcal{D}(\mathbb{R}^{1+n}) \text{ for } i = 1, 2, 3\}$ is dense in $\mathcal{D}(\mathbb{R}^{1+n})$, (3.4) will follow once it is proved that

$$[(N_\sigma * S_\sigma) * (e_{-\sigma}N)] * [\varphi_1 * \varphi_2 * \varphi_3] = [N_\sigma * (S_\sigma * (e_{-\sigma}N))] * [\varphi_1 * \varphi_2 * \varphi_3] \tag{3.2}$$

for every $\varphi_1, \varphi_2, \varphi_3 \in \mathcal{D}(\mathbb{R}^{1+n})$. In order to prove (3.2), fix φ_1, φ_2, φ_3 and let

$$f = N_\sigma * \varphi_1, \quad g = S_\sigma * \varphi_2, \quad h = (e_{-\sigma}U) * \varphi_3.$$

Then $f, g, h \in C^\infty(\mathbb{R}^{1+n})$ and using commutativity and associativity of convolution of distributions when all factors except at most one have compact support, one can prove that

$$[(N_\sigma * S_\sigma) * (e_{-\sigma}U)] * [\varphi_1 * \varphi_2 * \varphi_3] = (f * g) * h \tag{3.3}$$

and

$$[N_\sigma * (S_\sigma * (e_{-\sigma}U))] * [\varphi_1 * \varphi_2 * \varphi_3] = f * (g * h). \tag{3.4}$$

Let us stress that in the proof of (3.3) and (3.4) (and in particular in the proof that the right sides of (3.3) and (3.4) make sense) we have to make use of the facts that $N_\sigma * S_\sigma = \delta$ and $S_\sigma * (e_{-\sigma}U)$ has compact support. The equalities (3.3) and (3.4) reduce the problem of proving (3.2) to proving the equality

$$(f * g) * h = f * (g * h). \tag{3.5}$$

To do this, we need some more detailed information about f, g, h. Since $N_\sigma \in \mathcal{O}'_C(\mathbb{R}^{1+n})$, by (1.1) one has

$$f \in \mathcal{S}(\mathbb{R}^{1+n}) \subset L^1(\mathbb{R}^{1+n}). \tag{3.6}$$

Since $\operatorname{supp} S_\sigma = \{0\}$, one has

$$g \in \mathcal{D}(\mathbb{R}^{1+n}) \subset L^1(\mathbb{R}^{1+n}). \tag{3.7}$$

Furthermore

$$h \in C^\infty(\mathbb{R}; \mathcal{S}(\mathbb{R}^n)) \subset C(\mathbb{R}; L^1(\mathbb{R}^n)). \tag{3.8}$$

Indeed, for the proof of (3.8) it is sufficient to show that $[(e_{-\sigma}U) * \varphi_3]|_{[-a,a]\times\mathbb{R}^n} \in C^\infty([-a,a]; \mathcal{S}(\mathbb{R}^n))$ for every $a \in \,]0,\infty[$. So, take $a \in \,]0,\infty[$ and $b \in \,]0,\infty[$ such that $\operatorname{supp} \varphi_3 \subset [-b,b] \times \mathbb{R}^n$. Take $\vartheta \in \mathcal{D}(\mathbb{R})$ such that $\vartheta = 1$ on $[-a-b, a+b]$. Then

$$[(e_{-\sigma}U) * \varphi_3]|_{[-a,a]\times\mathbb{R}^n} = [(\vartheta_0 e_{-\sigma}U) * \varphi_3]|_{[-a,a]\times\mathbb{R}^n}. \tag{3.9}$$

Since $\vartheta_0 e_{-\sigma}U \in \mathcal{O}'_C(\mathbb{R}^{1+n})$, by (1.1) one has $(\vartheta e_{-\sigma}U) * \varphi_3 \in \mathcal{S}(\mathbb{R}^{1+n})$, so that (3.9) implies (3.8).

Since $\operatorname{supp} N_\sigma, \operatorname{supp} e_{-\sigma}U \subset H_+$ there is $c \in \,]0,\infty[$ (depending on φ_1, φ_2, φ_3, which however are fixed) such that

$$\operatorname{supp} f, \operatorname{supp} g, \operatorname{supp} h \subset \{(x_0, x_1, \ldots, x_n) \in \mathbb{R}^{1+n} : x_0 \geq -c\}. \tag{3.10}$$

From (3.6)–(3.8) and (3.10) it follows that $(|f| * |g|) * |h| \in C([-3c,\infty[; L^1(\mathbb{R}^n))$. Hence

$$\int_{\mathbb{R}^{1+n}} \left(\int_{\mathbb{R}^{1+n}} |f(v_0,\ldots,v_n)|\, |g(u_0-v_0,\ldots,u_n-v_n)|\, dv_0 \ldots dv_n \right) \cdot |h(x_0-u_0,\ldots,x_n-u_n)|\, du_0 \ldots du_n < \infty$$

for every $(x_0,\ldots,x_n) \in \mathbb{R}^{1+n}$, so that, by the Fubini–Tonelli theorem, the two iterated integrals corresponding to the integral

$$\int_{\mathbb{R}^{1+n}\times\mathbb{R}^{1+n}} f(v_0,\ldots,v_n) g(u_0-v_0,\ldots,u_n-v_n) h(x_0-u_0,\ldots,x_n-v_n) \, dv_0 \ldots dv_n \, du_0 \ldots du_n$$

are equal for every $(x_0,\ldots,x_n) \in \mathbb{R}^{1+n}$. This means that (3.5) holds.

3.2 Uniqueness as a consequence of the associativity relation (3.1)

The uniqueness of the fundamental solution belonging to $\mathcal{O}'_{\mathrm{LOC}}(H_+)$ for the operator $P(\partial_0, \partial_1, \ldots, \partial_n)$ satysfying (i) is a consequence of the following lemma.

Lemma 3.2. *Suppose that* (i) *holds and that* $F \in \mathcal{E}'(\mathbb{R}^{1+n})$ *has support contained in* H_+. *Then the equation*

$$P(\partial_0, \partial_1, \dots, \partial_n)U = F \tag{3.11}$$

has exactly one solution U *belonging to* $\mathcal{O}'_{\mathrm{LOC}}(H_+)$. *Moreover, for this solution and every* $\sigma \in \,]\omega_0, \infty[$ *one has*

$$U = (e_\sigma N_\sigma) * F. \tag{3.12}$$

Proof. Suppose that (i) holds. Take $\sigma \in \,]\omega_0, \infty[$. Then, in view of (2.18) and (2.17), $N = e_\sigma N_\sigma$ belongs to $\mathcal{O}'_{\mathrm{LOC}}(H_+)$ and is a fundamental solution for $P(\partial_0, \partial_1, \dots, \partial_n)$. It follows that U defined by (3.12) belongs to $\mathcal{O}'_{\mathrm{LOC}}(H_+)$ and satisfies (3.11). It remains to prove that in $\mathcal{O}'_{\mathrm{LOC}}(H_+)$ there are no other solutions of (3.11). To this end suppose that $U \in \mathcal{O}'_{\mathrm{LOC}}(H_+)$ and U satisfies (3.11). Take $\sigma \in \,]\omega_0, \infty[$ and define S_σ by (2.14). Then, by (2.15),

$$S_\sigma * (e_{-\sigma}U) = P(\sigma + \partial_0, \partial_1, \dots, \partial_n)(e_{-\sigma}U) = e_{-\sigma}(P(\partial_0, \partial_1, \dots, \partial_n)U) = e_{-\sigma}F,$$

whence, by (2.14), (2.6) and (3.1),

$$\begin{aligned} e_{-\sigma}U &= \delta * (e_{-\sigma}U) = (N_\sigma * S_\sigma) * (e_{-\sigma}U) = N_\sigma * (S_\sigma * (e_{-\sigma}U)) \\ &= N_\sigma * (e_{-\sigma}F) = e_{-\sigma}((e_\sigma N_\sigma) * F) \end{aligned}$$

so that $U = (e_\sigma N_\sigma) * F$.

4 Proof of (ii)⇒(i)

4.1 The distributions $\vartheta_0 N(\varphi \otimes \cdot)$

Let $N \in \mathcal{O}'_{\mathrm{LOC}}(H_+)$ be a fundamental solution for $P(\partial_0, \partial_1, \dots, \partial_n)$. Fix a, b such that $0 < a < b < \infty$, and $\vartheta \in \mathcal{D}(\mathbb{R})$ such that $\vartheta = 1$ on $[-b, b]$. For every $\varphi \in \mathcal{D}(\mathbb{R})$ consider the mapping

$$T(\varphi) : \mathcal{D}(\mathbb{R}^n) \ni \phi \mapsto \langle \vartheta_0 N, \varphi \otimes \phi \rangle \in \mathbb{C}.$$

Then $T(\varphi) \in \mathcal{D}'(\mathbb{R}^n)$. Since $\vartheta_0 N \in \mathcal{O}'_C(\mathbb{R}^{1+n})$, from (1.2) it follows that for every $k \in \mathbb{N}_0$ there is $m_k \in \mathbb{N}_0$ such that

$$\vartheta_0 N = \sum_{p+|\alpha| \le m_k} \partial_0^p \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n} F_{k;p,\alpha} \tag{4.1}$$

where every $F_{k;p,\alpha}$ is a continuous function on $\mathbb{R}^{1+n} = \{(t, x) \in \mathbb{R} \times \mathbb{R}^n\}$ for which

$$\sup_{(t,x) \in \mathbb{R}^{1+n}} (1 + |t| + |x|)^k |F_{k;p,\alpha}(t, x)| < \infty.$$

Consequently, whenever $\varphi \in \mathcal{D}(\mathbb{R})$, then

$$T(\varphi) = \sum_{|\alpha| \le m_k} \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n} f_{k;\alpha;\varphi} \tag{4.2}$$

where

$$f_{k;\alpha;\varphi}(x) = \sum_{p \le m_k - |\alpha|} \int_{\mathbb{R}} ((-\partial_0)^p \varphi(t)) F_{k;p;\alpha}(t,x)\, dt.$$

It follows that, whenever $|\alpha| \le m_k$, $\varphi \in \mathcal{D}(\mathbb{R})$, and $x \in \mathbb{R}^n$, then

$$\begin{aligned} |f_{k;\alpha;\varphi}(x)| &\le C_k \sum_{p \le m_k - |\alpha|} \int_{\operatorname{supp} \vartheta} |\partial_0^p \varphi(t)| (1 + |t| + |x|)^{-k}\, dt \\ &\le D_k (1 + |x|)^{-k} \sup\{|\partial_0^p \varphi(t)| : p = 0, \ldots, m_k,\ t \in \mathbb{R}\}, \end{aligned} \tag{4.3}$$

where $C_k, D_k \in \,]0, \infty[$ depend only on k. In particular this shows that

$$T(\varphi) \in \mathcal{O}'_C(\mathbb{R}^n) \quad \text{for every } \varphi \in \mathcal{D}(\mathbb{R}).^{*)} \tag{4.4}$$

Since N is the fundamental solution for $P(\partial_0, \partial_1, \ldots, \partial_n)$ with support in H_+, and $\vartheta = 1$ on $[-b, b]$, it follows that

(4.5) $T(\varphi) = 0$ whenever $\operatorname{supp} \varphi \subset \,]-\infty, 0[$,

(4.6) $\sum_{k=0}^{m} Q_k(\partial_1, \ldots, \partial_n) T((-\partial_0)^k \varphi) = \varphi(0)\delta$ for all $\varphi \in C^\infty_{[-b,b]}(\mathbb{R})$ where δ is the Dirac distribution on $\mathbb{R}^n$ and $Q_k(\partial_1, \ldots, \partial_n)$, $k = 0, \ldots, m$, are PDOs on $\mathbb{R}^n$ such that $P(\partial_0, \partial_1, \ldots, \partial_n) = \sum_{k=0}^{m} \partial_0^k Q_k(\partial_1, \ldots, \partial_n)$.

In the subsequent lemmas it will be tacitly assumed that (ii) holds and N, a, b, ϑ, T are fixed. Recall that $0 < a < b < \infty$, $\vartheta \in \mathcal{D}(\mathbb{R})$, $\vartheta = 1$ on $[-b, b]$, $N \in \mathcal{O}'_{\mathrm{LOC}}(H_+)$ is a fundamental solution for $P(\partial_0, \partial_1, \ldots, \partial_n)$ and $T(\varphi) = \vartheta_0 N(\varphi \otimes \cdot) \in \mathcal{O}'_C(\mathbb{R}^n)$ for every $\varphi \in \mathcal{D}(\mathbb{R})$. For every $\varphi \in \mathcal{D}(\mathbb{R})$ denote by $\widehat{T}(\varphi)$ the image of $T(\varphi)$ under the Fourier transformation on $\mathbb{R}^n$. Then $\widehat{T}(\varphi) \in \mathcal{O}_M(\mathbb{R}^n)$, by (4.4) and (1.4).

Lemma 4.1. *There are $p_0, m_0 \in \mathbb{N}_0$ and $C \in \,]0, \infty[$ such that*

$$|\widehat{T}(\varphi)(\xi)| \le C(1 + |\xi|)^{m_0} \sup\{|\partial_0^p \varphi(t)| : p = 0, \ldots, p_0,\ a \le t \le b\}$$

for every $\xi \in \mathbb{R}^n$ and $\varphi \in C^\infty_{[a,b]}(\mathbb{R})$.

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*) After introducing the topology in $\mathcal{O}'_C(\mathbb{R}^n)$, it is possible to prove that the mapping $\mathcal{D}(\mathbb{R}) \ni \varphi \mapsto T(\varphi) \in \mathcal{O}'_C(\mathbb{R}^n)$ is a vector-valued distribution. However this is insignificant for the present proof.

Proof. If in (4.1) we take $k > n$, then, by (4.2) and (4.3),

$$T(\varphi) = \sum_{|\alpha| \le m_k} \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n} f_{k;\alpha;\varphi} \quad \text{for every } \varphi \in C^\infty_{[a,b]}(\mathbb{R})$$

where

$$\|f_{k;\alpha;\varphi}\|_{L^1(\mathbb{R}^n)} \le D \sup\{|\partial_0^p \varphi(t)| : p = 0, \dots, m_k,\ a \le t \le b\}$$

for every α with $|\alpha| \le m_k$ and every $\varphi \in C^\infty_{[a,b]}(\mathbb{R})$, with $D \in\,]0, \infty[$ depending only on k. Consequently, whenever $\varphi \in C^\infty_{[a,b]}(\mathbb{R})$, then

$$|\widehat{T}(\varphi)(\xi)| \le (1 + |\xi|)^{m_k} |g_\varphi(\xi)|_{M_{m\times m}} \quad \text{for every } \xi \in \mathbb{R}^n$$

where $g_\varphi \in C_b(\mathbb{R}^n)$ and

$$\sup_{\xi \in \mathbb{R}^n} |g_\varphi(\xi)| \le C \sup\{|\partial_0^p \varphi(t)| : p = 0, \dots, m_k,\ a \le t \le b\}$$

for some $C \in\,]0, \infty[$ depending only on k.

4.2 An inequality of Chazarain type

Lemma 4.2. *Suppose that $P(\partial_0, \partial_1, \dots, \partial_n)$ is a PDO on $\mathbb{R}^{1+n}$ with constant coefficients for which there is a fundamental solution belonging to $\mathcal{O}'_{\mathrm{LOC}}(H_+)$. Then there are $a, b \in\,]0, \infty[$ such that whenever $(\lambda, \xi) \in \mathbb{C} \times \mathbb{R}^n$ and*

$$\operatorname{Re} \lambda > a + b \log(1 + |\lambda| + |\xi|)\ ^{*)},$$

then $P(\lambda, i\xi) \ne 0$.

Proof. From (4.6) it follows that

$$\sum_{k=0}^{m} Q_k(i\xi) \widehat{T}((-\partial_0)^k \varphi) = \varphi(0) \tag{4.7}$$

for every $\varphi \in C^\infty_{[-b,b]}(\mathbb{R})$ and $\xi \in \mathbb{R}^n$. Take $\varphi_0 \in C^\infty_{[-b,b]}(\mathbb{R})$ such that $\varphi_0 = 1$ on $[-a, a]$. Following J. Chazarain [Cha], pp. 394–395, consider functions

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$^{*)}$ This inequality and its proof are similar to the inequality (1.2) on p. 394 of [Cha] and the argument presented on p. 395 of [Cha]. There is however an important difference. In [Cha] the inequality (1.2) does not involve ξ and determines the “logarithmic region” $\Lambda \subset \mathbb{C}$ such that for every $\lambda \in \Lambda$ an abstract operator $Q(\lambda) = \lambda^m A_m + \cdots + \lambda A_1 + A_0$ is invertible. In our case the inequality involves ξ but the operator $Q(\lambda)$ is replaced by the *polynomial* $P(\lambda, i\xi)$, and Lemma 4.2 is not the final step of the argument.

of the form $\varphi = e_{-\lambda}\varphi_0$ where λ ranges over $\mathbb{C}$. Since $T(\varphi) = 0$ whenever $\operatorname{supp}\varphi \subset]-\infty, 0[$, by (4.7) and the Leibniz formula one has

$$(4.8)\qquad \begin{aligned} P(\lambda, i\xi)[\widehat{T}(e_{-\lambda}\varphi_0)](\xi) &= \Big(\sum_{k=0}^{m} \lambda^k Q_k(i\xi)\Big)[\widehat{T}(e_{-\lambda}\varphi_0)](\xi) \\ &= 1 - \sum_{k=0}^{m} Q_k(i\xi)[\widehat{T}(\psi_{k,\lambda})](\xi) \end{aligned}$$

where $\psi_{k,\lambda} \in C^\infty_{[a,b]}(\mathbb{R})$ is determined by the equality

$$\psi_{k,\lambda}(t) = (-1)^k \sum_{j=1}^{k} \binom{k}{j} \partial_0^j \varphi_0(t)(-\lambda)^{k-j} e^{-\lambda t} \quad \text{for } t \in [a, b].$$

By Lemma 4.1 there are $C, K \in]0, \infty[$ such that, if $\operatorname{Re}\lambda \geq 0$, then

$$(4.9)\qquad \begin{aligned} &|[\widehat{T}(\psi_{k,\lambda})](\xi)| \\ &\quad \leq C(1 + |\xi|)^{m_0} \sup\{|\partial_0^p \psi_{k,\lambda}(t)| : p = 0, \dots, p_0,\ a \leq t \leq b\} \\ &\quad \leq C(1 + |\xi|)^{m_0} K(1 + |\lambda|)^{m-1+p_0} e^{-a\operatorname{Re}\lambda} \end{aligned}$$

for every $k = 1, \dots, m$ and $\xi \in \mathbb{R}^n$. Furthermore, there are $l \in \mathbb{N}$ and $L \in]0, \infty[$ such that

$$(4.10)\qquad \sum_{k=0}^{m} |Q_k(i\xi)| \leq L(1 + |\xi|)^l \quad \text{for every } \xi \in \mathbb{R}^n.$$

Let $\mu = m_0 + l + m - 1 + p_0$. From (4.8)–(4.10) it follows that if $(\lambda, \xi) \in \mathbb{C} \times \mathbb{R}^n$, $\operatorname{Re}\lambda \geq 0$, and

$$CKL(1 + |\lambda| + |\xi|)^\mu e^{-a\operatorname{Re}\lambda} < 1,$$

then $|\sum_{k=0}^{m} Q_k(i\xi)[\widehat{T}(\psi_{k,\lambda})](\xi)| < 1$, and hence $P(\lambda, i\xi) \neq 0$. Therefore, if $(\lambda, \xi) \in \mathbb{C} \times \mathbb{R}^n$ and

$$\operatorname{Re}\lambda > a^{-1}\log(CKL + 1) + a^{-1}\mu\log(1 + |\lambda| + |\xi|),$$

then $P(\lambda, i\xi) \neq 0$.

4.3 The Chazarain type inequality implies (i)

The implication (ii)⇒(i) is an immediate consequence of Lemma 4.2 and the following

Lemma 4.3. *Let Q be a polynomial of $1+n$ variables with complex coefficients. Suppose that there are $a \in \mathbb{R}$ and $b \in]0,\infty[$ such that*

$$\operatorname{Re}\lambda \le a + b\log(1+|\lambda|+|\xi|) \quad \text{whenever } (\lambda,\xi) \in \mathbb{C}\times\mathbb{R}^n \text{ and } Q(\lambda,\xi)=0. \tag{4.11}$$

Then

$$\sup\{\operatorname{Re}\lambda : (\lambda,\xi)\in\mathbb{C}\times\mathbb{R}^n,\ Q(\lambda,\xi)=0\} < \infty.$$

The proof follows the scheme due to L. Gårding and L. Hörmander. Let

$$\sigma(r) = \sup\{\operatorname{Re}\lambda : \lambda\in\mathbb{C} \text{ and there is } \xi\in\mathbb{R}^n \text{ such that } |\lambda^2|+|\xi^2| \le \tfrac{1}{2}r^2 \text{ and } Q(\lambda,\xi)=0\}.$$

Then, by (4.11),

$$\sigma(r) \le a + b\log(1+r) \quad \text{for every } r\in[0,\infty[. \tag{4.12}$$

Following an idea of L. Hörmander (presented in the Appendix to [H2]), the Tarski–Seidenberg theorem is used to show that there is a polynomial $V(z,w)$ (not vanishing identically) of two variables such that $V(r,\sigma(r))=0$ for every $r\in[0,\infty[$. Then, as in L. Gårding's proof of the Lemma on p. 11 of [G], the Puiseux expansions of the w-roots of $V(z,w)$ for large $|z|$ show that (4.12) is possible only if $\sup\{\sigma(r) : r\in[0,\infty[\} < \infty$.

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Appendix

Proof of Lemma 4.3

Let $R(\sigma,\tau,\xi)$ and $S(\sigma,\tau,\xi)$ be real polynomials on $\mathbb{R}^{2+n}$ such that

$$R(\sigma,\tau,\xi)+iS(\sigma,\tau,\xi)=Q(\sigma+i\tau,\xi).$$

Then

$$E=\{(r,\sigma,\tau,\xi)\in\mathbb{R}^{3+n}: r\geq 0,\ \sigma^2+\tau^2+|\xi|^2\leq \tfrac{1}{2}r^2,\\ R(\sigma,\tau,\xi)=0,\ S(\sigma,\tau,\xi)=0\}$$

is a semi-algebraic subset of $\mathbb{R}^{3+n}$ and, by the Tarski–Seidenberg theorem (see Appendix to [H2]) its projection on $\mathbb{R}^2$ defined by

$$F=\{(r,\sigma)\in\mathbb{R}^2: \exists_{\tau,\xi}\ (r,\sigma,\tau,\xi)\in E\}$$

is a semi-algebraic subset of $\mathbb{R}^2$. If $\sigma(r)$ is defined as in Sec. 4.3, then for every $r\in[0,\infty[$ one has

$$\sigma(r)=\sup\{\sigma:(r,\sigma)\in F\}. \tag{A.1}$$

Since F is semi-algebraic, it may be represented in the form

$$F=\bigcup_{i=1}^{k}F_i\cap G_{i,1}\cap\cdots\cap G_{i,j(i)} \tag{A.2}$$

where

$$F_i=\{x\in\mathbb{R}^2: P_i(x)=0\},\qquad G_{i,j}=\{x\in\mathbb{R}^2: Q_{i,j}(x)>0\}, \tag{A.3}$$

P_i and $Q_{i,j}$ being real polynomials on $\mathbb{R}^2$. It is not excluded that some P_i are identically zero and some $Q_{i,j}$ are strictly positive on the whole $\mathbb{R}^2$. From (A.1) it follows that whenever $r\in[0,\infty[$ is fixed, there is $i(r)\in\{1,\ldots,k\}$ such that

$$\sigma(r)=\sup\{\sigma:(r,\sigma)\in F_{i(r)}\cap G_{i(r),1}\cap\cdots\cap G_{i(r),j(i(r))}\}. \tag{A.4}$$

By (4.12), for every $r\in[0,\infty[$ one has $\sigma(r)<\infty$, so that there is a bounded sequence $(\sigma_\nu(r))_{\nu=1}^{\infty}$ such that

$$(r,\sigma_\nu(r))\in F_{i(r)}\cap G_{i(r),1}\cap\cdots\cap G_{i(r),j(i(r))}\qquad\text{for every } \nu=1,2,\ldots \tag{A.5}$$

and

$$\lim_{\nu\to\infty}\sigma_\nu(r)=\sigma(r). \tag{A.6}$$

If $P_{i(r)}\not\equiv 0$, then (A.5) and (A.6) imply that $P_{i(r)}(r,\sigma(r))=0$. If $P_{i(r)}\equiv 0$, then, again by (A.5) and (A.6), for some $j_0\in\{1,\dots,j(i(r))\}$ one has $Q_{i(r),j_0}\not\equiv 0$ and $Q_{i(r),j_0}(r,\sigma(r))=0$, because otherwise $F_{i(r)}=\mathbb{R}^2$ and there would be $\varepsilon>0$ such that $Q_{i(r),j}(r,\sigma(r)+\varepsilon)>0$ for every $j\in\{1,\dots,j(i(r))\}$ contrary to (A.4). Consequently, whenever $r\in[0,\infty[$, then either $W_r\equiv P_{i(r)}$ or $W_r\equiv Q_{i(r),j_0}$ is a real polynomial on $\mathbb{R}^2$ such that

$$W_r\not\equiv 0\quad\text{and}\quad W_r(r,\sigma(r))=0.$$

Therefore if V is equal to the product of all those polynomials P_i and $Q_{i,j}$, that occur in (A.3) and do not vanish identically on $\mathbb{R}^2$, then

$$V\not\equiv 0\quad\text{and}\quad V(r,\sigma(r))=0\quad\text{for every } r\in[0,\infty[. \tag{A.7}$$

Now we are going to show that (4.12) and (A.7) imply $\sup\{\sigma(r):r\in[0,\infty[\}<\infty$. To this end we consider V as a polynomial $V(z,w)$ of two complex veriables, and, following L. Gårding [G, proof of the Lemma on p. 11], we use the Puiseux expansions of the w-roots of $V(z,w)$. Concerning these expansions we will give exact references to [S-Z]. Consider the factorization

$$V(z,w)=V_1(z,w)\cdot V_2(z,w)\cdot\ldots\cdot V_l(z,w),\quad z\in\mathbb{C}\setminus\bigcup_{k=1}^{l}S_k,\quad x\in\mathbb{C},$$

where

(i) every V_k, $k=1,\dots,l$, belongs to the ring $K(z)[w]$ of polynomials of w over the field $K(z)$ of rational functions of z, so that

$$V_k(z,w)=\sum_{j=0}^{d_k}A_{k,j}(z)w^j\quad\text{ for every } z\in\mathbb{C}\setminus S_k \text{ and } x\in\mathbb{C}$$

where $A_{k,j}\in K(z)$ for $j=0,\dots,d_k$, $A_{k,d_k}\not\equiv 0$, and the finite set S_k consists of those points of $\mathbb{C}$ at which some $A_{k,j}$, $j=0,\dots,d_k$, has a pole,

(ii) every V_k, $k=1,\dots,l$, is an irreducible element of $K(z)[w]$.

The assumption that $A_{k,d_k}\not\equiv 0$ implies that all the sets

$$N_k=\{z\in\mathbb{C}\setminus S_k : A_{k,d_k}(z)=0\},\quad k=1,\dots,l,$$

are finite. Define

$$\begin{aligned} M_k &= \{z \in \mathbb{C} \setminus (S_k \cup N_k) : \text{not all the } w\text{-roots of } V_k(z,w) \text{ are simple}\}, \\ \mathcal{N}_k &= \{(z,w) \in (\mathbb{C} \setminus (S_k \cup N_k \cup M_k)) \times \mathbb{C} : V_k(z,w) = 0\}. \end{aligned}$$

From Theorems VI.13.7, VI.14.2 and VI.14.3 of [S-Z] it follows that

(a) for every $k = 1, \dots, l$ the set M_k is finite and

$$\mathcal{N}_k \cap [(\mathbb{C} \setminus (S_k \cup N_k \cup M_k)) \times \mathbb{C}]$$

is equal to the graph of a d_k-variate function $\mathcal{R}_k$ analytic on the set $\mathbb{C} \setminus (S_k \cup N_k \cup M_k)$,

(b) there is $R \in]0, \infty[$ such that for every $k = 1, \dots, l$ one has $\{z \in \mathbb{C} : R < |z| < \infty\} \subset \mathbb{C} \setminus (S_k \cup N_k \cup M_k)$, and if $z \in \mathbb{C}$ and $R < |z| < \infty$, then

$$\mathcal{R}_k(z) = \{\phi_k(\zeta) : \zeta \in \mathbb{C},\ 0 < |\zeta| < R^{-1/d_k},\ \zeta^{d_k} = z^{-1}\}$$

where ϕ_k is a function of one complex variable holomorphic in the annulus $\{\zeta \in \mathbb{C} : 0 < |\zeta| < R^{-1/d_k}\}$,

(c) every ϕ_k, $k = 1, \dots, l$, has at zero either a removable singularity or a pole.

Consequently, for every $k = 1, \dots, l$ one has

$$\text{(A.8)} \quad \mathcal{N}_k \cap (\{z \in \mathbb{C} : |z| > R\} \times \mathbb{C}) = \Big\{\Big(z, \sum_{p=p_k}^{\infty} a_{k,p}\zeta^p\Big) : (z,\zeta) \in \mathbb{C}^2,\ |z| > R,\ \zeta^{d_k} = z^{-1}\Big\}$$

where $\sum_{p=p_k}^{\infty} a_{k,p}\zeta^p$ is the Laurent expansion of ϕ_k in the annulus $\{\zeta \in \mathbb{C} : 0 < |\zeta| < R^{-1/d_k}\}$. We assume that either $a_{k,p_k} \neq 0$ or $0 = a_{k,p_k} = a_{k,p_k+1} = \cdots$. The equality (A.8) is nothing but the exact form of the Puiseux series expansion of $\mathcal{R}_k(z)$ for $z \to \infty$. It follows that if $r \in]R, \infty[$, then $(r, \sigma(r)) \subset \bigcup_{k=1}^{l} \mathcal{N}_k$ and $\sigma(r)$ is equal to one of the numbers

$$\sigma_{k,d}(r) = \sum_{p=p_k}^{\infty} a_{k,p}\left(\frac{e^{i2\pi d/d_k}}{\sqrt[d_k]{r}}\right)^p, \qquad k = 1, \dots, l,\ d = 1, \dots, d_k,$$

where $\sqrt[d_k]{r_k}$ is the positive d_k-th root of r and the series is absolutely convergent, so that $\sigma_{k,d}(r) = c_{k,d} r^{-p_k/d_k}(1 + o(1))$ as $r \to \infty$ where $c_{k,d} = a_{k,p_k} e^{i2\pi d p_k/d_k}$. If for some $k = 1, \dots, l$ and $d = 1, \dots, d_k$ the set

$$\{r \in]R, \infty[: \sigma(r) = \sigma_{k,d}(r)\}$$

is unbounded, then $c_{k,d}$ must be real, and, by the estimation (4.12) of $\sigma(r)$, either $c_{k,d} \leq 0$, or $c_{k,d} > 0$ and $p_k \geq 0$. In both cases $\sup\{\sigma_{k,d}(r) : r \in]R,\infty[\} < \infty$. This implies that $\sup\{\sigma(r) : r \in [0,\infty[\} < \infty$, completing the proof.

References

[S-Z] S. Saks and A. Zygmund, *Analytic Functions*, 3rd ed., PWN, Warszawa, 1959 (in Polish); English transl.: PWN, 1965; French transl.: Masson, 1970.